ON THE GENERALIZED CLIMBING STAIRS PROBLEM

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ABSTRACT. Let $\mathcal{S}$ be a subset of the positive integers, and M be a positive integer. Mohammad K. Azarian, inspired by work of Tony Colledge, considered the number of ways to climb a staircase containing n stairs using "step-sizes" $s \in \mathcal{S}$ and multiplicities at most M.

In this exposition, we find a solution via generating functions, i.e., an expression which counts the number of partitions $n = \sum_{s \in \mathcal{S}} m_s \, s$ satisfying $0 \leq m_s \leq M$. We then use this result to answer a series of questions posed by Azarian, thereby showing a link with ten sequences listed in the On-Line Encyclopedia of Integer Sequences. We conclude by posing open questions which seek to count the number of compositions of n.

1. INTRODUCTION

"I'll build a stairway to paradise / with a new step ev'ry day."
– Ira and George Gershwin, *Scandals* (1922)

Consider the problem of determining the number of ways to climb a staircase containing n stairs. For example, there are three ways to climb $n = 3$ stairs: take one stair three times, take one stair then two stairs, or take three stairs in one step. It is natural to restrict this problem by considering the number of ways to climb n stairs where the types of steps taken are either even-numbered or odd-numbered. For example, there are just two ways to climb a staircase containing $n = 3$ stairs if only odd-numbered steps are allowed. One may further restrict this problem by considering only a distinct set of steps. For example, there are three such ways to climb a staircase containing $n = 3$ stairs: take one stair then two stairs, or take three stairs in just one step. One may also consider the cases when the order of the steps taken is relevant.

Azarian [2], [3], inspired by Colledge [5], considered problems which can be phrased as follows:

Generalized Climbing Stairs Problem. *Let $\mathcal{S}$ be a subset of the positive integers; this will denote the "sizes" of steps allowed. Let M be a positive integer; this will denote the maximum multiplicity of each step-size $s \in \mathcal{S}$ taken. What are*

the possible ways to climb a staircase containing n stairs taking step-sizes $\mathcal{S}$ and multiplicities at most M?

The main focus of this exposition is to give a complete answer to this problem. In Section 2, we find a solution via generating functions, i.e., an expression which counts the number of partitions $n = \sum_{s \in \mathcal{S}} m_s\, s$ satisfying $0 \leq m_s \leq M$. In Section 3, we use this result to answer a series of questions posed by Azarian, thereby showing a link with ten sequences listed in the On-Line Encyclopedia of Integer Sequences. In Section 4, we conclude by posing open questions which seek to count the number of compositions of n.

2. Main Theorem

First, we fix some notation. Let $\mathcal{S}$ be a subset of the positive integers; this will denote the "sizes" of steps taken when climbing the stairs. For example, $\mathcal{S}$ can be the set of all even positive integers, the set of all odd positive integers, or perhaps a finite set. Assume that we have a staircase containing n stairs. We denote a set of "steps" of this staircase by the $\mathcal{S}$-partition

$$n = \sum_{s \in \mathcal{S}} m_s\, s$$

where s is the "step-size" and m_s denotes the multiplicity. We will also denote this $\mathcal{S}$-partition by the sequence

$$(m_s)_{s \in \mathcal{S}}$$

where m_s are nonnegative integers, all but finitely many of which are zero. For example, there are three ways to climb a $n = 3$ stairs: take one stair three times, take one stair then two stairs, or take three stairs at once in one step. If we let $\mathcal{S}$ denote the set of all positive integers, then we may express these three ways as $(3, 0, 0, 0, \dots)$, $(1, 1, 0, 0, \dots)$, or $(0, 0, 1, 0, \dots)$, respectively. For a positive integer M, denote $p_{\mathcal{S}}^{(M)}(n)$ as the number of ways to climb a staircase containing n steps using step-sizes $\mathcal{S}$ and multiplicities at most M. To be precise,

$$p_{\mathcal{S}}^{(M)}(n) \quad = \quad \# \left\{ (m_s)_{s \in \mathcal{S}} \;\middle|\; n = \sum_{s \in \mathcal{S}} m_s\, s \text{ and } 0 \leq m_s \leq M \right\}.$$

We use the convention $p_{\mathcal{S}}^{(M)}(0) = 1$.

Azarian [2], [3], inspired by Colledge [5], considered problems which can be phrased as follows:

Generalized Climbing Stairs Problem. *Let $\mathcal{S}$ be a subset of the positive integers; this will denote the "sizes" of steps allowed. Let M be a positive integer; this will denote the maximum multiplicity of each step-size $s \in \mathcal{S}$ taken. What are*

the possible ways to climb a staircase containing n stairs taking step-sizes $\mathcal{S}$ and multiplicities at most M?

In this case order is irrelevant. It is easy see that climbing a staircase gives a $\mathcal{S}$-partition of n. We prove the following as a solution to the problem above:

Theorem. *Let $p_{\mathcal{S}}^{(M)}(n)$ denote the number of ways to climb a staircase containing n stairs using step-sizes $\mathcal{S}$ and multiplicities at most M. Then we have the identity*

$$\sum_{n=0}^{\infty} p_{\mathcal{S}}^{(M)}(n)\, x^n = \prod_{s \in \mathcal{S}} \frac{1 - x^{(M+1)s}}{1 - x^s} \quad \text{on the interval} \quad |x| < 1.$$

Remarks.

(a) When M is unbounded (i.e., "$M = \infty$") and $\mathcal{S}$ is the set of all positive integers, then $p_{\mathcal{S}}^{(M)}(n) = p(n)$ which is the classical partition function. This result is a generalization of the classical identity found by Euler:

$$\sum_{n=0}^{\infty} p(n)\, x^n = \prod_{s=1}^{\infty} \frac{1}{1 - x^s} = 1 + x + 2\,x^2 + 3\,x^3 + 5\,x^4 + 7\,x^5 + \cdots.$$

(b) When $M = 1$ and $\mathcal{S}$ is the set of all positive integers, then $p_{\mathcal{S}}^{(M)}(n) = q(n)$ counts the number of ways to climb the staircase using *distinct* step-sizes from $\mathcal{S}$:

$$\sum_{n=0}^{\infty} q(n)\, x^n = \prod_{s=1}^{\infty} (1 + x^s) = 1 + x + x^2 + 2\,x^3 + 2\,x^4 + 3\,x^5 + \cdots.$$

These sequences may be verified using the On-Line Encyclopedia of Integer Sequences [8]. Compare these with sequences A000041 and A000009, respectively.

Proof. We review [1, Theorem 1.1] and its proof to remind the reader of the language of climbing staircases. Fix real numbers r and T satisfying $0 < r < 1$ and $0 < T$. Let $\mathcal{S}_T \subseteq \mathcal{S}$ and let $s \in \mathcal{S}$ such that $s \leq T$. Define the function

$$G_T(x) = \prod_{s \in \mathcal{S}_T} \frac{1 - x^{(M+1)s}}{1 - x^s} \quad \text{on the interval} \quad |x| \leq r.$$

Since this function is a finite product of terms involving the geometric series, and the geometric series is uniformly convergent in this compact interval, we may rearrange this expression to yield

$$G_T(x) = \prod_{s \in \mathcal{S}_T} \frac{1 - x^{(M+1)s}}{1 - x^s}$$

$$= \prod_{s \in \mathcal{S}_T} \left(\frac{1}{1 - x^s} - x^{(M+1)s} \cdot \frac{1}{1 - x^s} \right)$$

$$= \prod_{s \in \mathcal{S}_T} \left(\sum_{m_s=0}^{\infty} x^{m_s s} - \sum_{m_s=0}^{\infty} x^{(m_s+M+1)s} \right)$$

$$= \prod_{s \in \mathcal{S}_T} \left(\sum_{m_s=0}^{M} x^{m_s s} \right).$$

By comparing the coefficient of x^n, we obtain the identity

$$G_T(x) = \prod_{s \in \mathcal{S}_T} \left(\sum_{m_s=0}^{M} x^{m_s s} \right)$$

$$= \sum_{n=0}^{\infty} \# \left\{ (m_s)_{s \in \mathcal{S}_T} \, \middle| \, n = \sum_{s \in \mathcal{S}_T} m_s \, s \text{ and } 0 \le m_s \le M \right\} x^n.$$

Since r and T were arbitrary, the theorem holds in the open interval $|x| < 1$ for any given set $\mathcal{S}$. $\qquad\square$

3. APPLICATIONS

We are now interested in applying the theorem in the previous section to counting the number of ways to climb a staircase. In this section, we answer questions of Mohammad Azarian originally posed in [3].

Question 1. *How many different ways are there to climb a staircase containing n stairs taking distinct even- (odd-, respectively) numbered step-sizes?*

Let $M = 1$ and $\mathcal{S}$ be the set of even (odd, respectively) positive integers. Then using the main theorem from the previous section, we have the generating function

$$\sum_{n=0}^{\infty} p_{\mathcal{S}}^{(M)}(n) \, x^n = \begin{cases} \prod_{s \text{ even}} (1 + x^s) = 1 + x^2 + x^4 + 2\,x^6 + \cdots \\ \qquad\qquad \text{for even-numbered steps;} \\ \\ \prod_{s \text{ odd}} (1 + x^s) = 1 + x + x^3 + x^4 + x^5 + \cdots \\ \qquad\qquad \text{for odd-numbered steps.} \end{cases}$$

For the former, compare with sequence A000009 of [8]. For the latter, compare with sequence A000700.

Question 2. *For a positive integer k, how many different ways are there to climb a staircase containing n stairs taking exactly k stairs at most k times?*

Denote $M = k$ and $\mathcal{S} = \{k\}$. Then we have the generating function

$$\sum_{n=0}^{\infty} p_{\mathcal{S}}^{(M)}(n)\, x^n = \frac{1 - x^{(k+1)k}}{1 - x^k} = 1 + x^k + x^{2k} + \cdots + x^{k^2}.$$

That is, $p_{\mathcal{S}}^{(M)}(n) \neq 0$ if and only if n is a multiple of k in the form $n = m\,k$ for some positive integer $m \leq k$.

Question 3. *How many different ways are there to climb a staircase containing n stairs taking at least two stairs at a time?*

Let M be unbounded and $\mathcal{S} = \{2,\, 3,\, 4,\, \dots\}$. Then we have the generating function

$$\sum_{n=0}^{\infty} p_{\mathcal{S}}^{(M)}(n)\, x^n = \prod_{s=2}^{\infty} \frac{1}{1 - x^s} = 1 + x^2 + x^3 + 2\,x^4 + 2\,x^5 + \cdots.$$

Compare with sequence A002865 of [8].

Question 4. *How many different ways are there to climb a staircase containing n stairs taking at most two stairs at a time?*

Let M be unbounded and $\mathcal{S} = \{1,\, 2\}$. Then we have the generating function

$$\sum_{n=0}^{\infty} p_{\mathcal{S}}^{(M)}(n)\, x^n = \frac{1}{1 - x} \cdot \frac{1}{1 - x^2}$$
$$= 1 + x + 2\,x^2 + 2\,x^3 + 3\,x^4 + 3\,x^5 + \cdots.$$

Compare with sequence A008619 of [8].

Question 5. *How many different ways are there to climb a staircase containing n stairs taking even- (odd-, respectively) numbered step-sizes?*

Let M be unbounded and $\mathcal{S}$ as the set of even (odd, respectively) positive integers. Then we have the generating function

$$\sum_{n=0}^{\infty} p_{\mathcal{S}}^{(M)}(n)\, x^n = \begin{cases} \displaystyle\prod_{s \text{ even}} \frac{1}{1 - x^s} = 1 + x^2 + 2\,x^4 + 3\,x^6 + \cdots \\ \qquad\qquad \text{for even-numbered steps;} \\[2ex] \displaystyle\prod_{s \text{ odd}} \frac{1}{1 - x^s} = 1 + x + x^2 + 2\,x^3 + 2\,x^4 + 3\,x^5 + \cdots \\ \qquad\qquad \text{for odd-numbered steps.} \end{cases}$$

For the former, compare with sequence A000041 of [8]. For the latter, compare with sequence A000009; it is well-known that $p_{\mathcal{S}}^{(M)}(n) = q(n)$ in this case. See [9] for more information.

Question 6. *How many different ways are there to climb a staircase containing n stairs where the multiplicity of each step-size is at most 2?*

Denote $M = 2$ and $\mathcal{S}$ as the set of positive integers. Then we have the generating function

$$
\begin{aligned}
\sum_{n=0}^{\infty} p_{\mathcal{S}}^{(M)}(n)\, x^n &= \prod_{s=1}^{\infty} \left(1 + x^s + x^{2s}\right) \\
&= 1 + x + 2\,x^2 + 2\,x^3 + 4\,x^4 + 5\,x^5 + \cdots .
\end{aligned}
$$

Compare with sequence A000726 of [8].

Question 7. *For a positive integer k, how many different ways are there to climb a staircase containing n stairs taking exactly k stairs for each step?*

Let M be unbounded and $\mathcal{S} = \{k\}$. Then we have the generating function

$$
\sum_{n=0}^{\infty} p_{\mathcal{S}}^{(M)}(n)\, x^n = \frac{1}{1 - x^k} = 1 + x^k + x^{2k} + x^{3k} + x^{4k} + x^{5k} + \cdots .
$$

That is, $p_{\mathcal{S}}^{(M)}(n) \neq 0$ if and only if n is a multiple of k.

Question 8. *How many different ways are there to climb a staircase containing n stairs where the size of each step is a prime number? What if the multiplicity of each step-size is at most 1?*

These questions were first considered by Bateman and Erdös [4] and Gupta [6], respectively. Let M be unbounded and $\mathcal{S} = \{2,\, 3,\, 5,\, 7,\, \ldots,\, \ell,\, \ldots\}$ denote the set of all prime numbers. Then we have the generating function

$$
\sum_{n=0}^{\infty} p_{\mathcal{S}}^{(M)}(n)\, x^n = \prod_{\ell \text{ prime}} \frac{1}{1 - x^\ell} = 1 + x^2 + x^3 + x^4 + 2\,x^5 + \cdots .
$$

Compare with sequence A000607 of [8]. Now let $M = 1$ with $\mathcal{S}$ as before. Then we have the generating function

$$
\sum_{n=0}^{\infty} p_{\mathcal{S}}^{(M)}(n)\, x^n = \prod_{\ell \text{ prime}} \left(1 + x^\ell\right) = 1 + x^2 + x^3 + 2\,x^5 + \cdots .
$$

Compare with sequence A000586 of [8].

Question 9. *How many different ways are there to climb a staircase containing n stairs where the size of each step is a Fibonacci number? What if the multiplicity of each step-size is at most 1?*

This question was first considered by Klarner [7]. Let M be unbounded and $\mathcal{S} = \{1, 2, 3, 5, \ldots, F_n, \ldots\}$ denote the set of all Fibonacci numbers, i.e., the Fibonacci sequence with multiplicities removed. Then we have the generating function

$$\sum_{n=0}^{\infty} p_{\mathcal{S}}^{(M)}(n)\, x^n = \prod_{F_n \text{ Fibonacci}} \frac{1}{1 - x^{F_n}}$$
$$= 1 + x + 2\,x^2 + 3\,x^3 + 4\,x^4 + 6\,x^5 + \cdots.$$

Compare with sequence A003107 of [8]. Now let $M = 1$ with $\mathcal{S}$ as before. Then we have the generating function

$$\sum_{n=0}^{\infty} p_{\mathcal{S}}^{(M)}(n)\, x^n = \prod_{F_n \text{ Fibonacci}} \left(1 + x^{F_n}\right)$$
$$= 1 + x + x^2 + 2\,x^3 + x^4 + 2\,x^5 + \cdots.$$

Compare with sequence A000119 of [8].

Question 10. *For positive integers a and b, how many different ways are there to climb a staircase containing n stairs where the size of each step s satisfies $a \le s \le b$?*

Let M be unbounded and $\mathcal{S}$ denote the set integers s satisfying $a \le s \le b$. Then we have the generating function

$$\sum_{n=0}^{\infty} p_{\mathcal{S}}^{(M)}(n)\, x^n = \prod_{s=a}^{b} \frac{1}{1 - x^s} = 1 + x^a + \cdots.$$

4. OPEN PROBLEM

Now consider the problem of determining the number of ways to climb a staircase containing n stairs – where the order in which the steps are taken is relevant. For example, there are *four* ways to climb $n = 3$ stairs: take one stair three times, take one stair then two stairs, take two stairs then one stair, or take three stairs at once in one step. We pose the following problem:

Generalized Climbing Stairs Problem with Order. *Let $\mathcal{S}$ be a subset of the positive integers; this will denote the "sizes" of steps allowed. Let M be a positive integer; this will denote the maximum multiplicity of each step-size $s \in \mathcal{S}$ taken. What are the possible ways to climb a staircase containing n stairs – keeping track of the order in which the steps are taken – if one can only take using step-sizes $\mathcal{S}$ and multiplicities at most M?*

When the order is irrelevant, climbing a staircase gives a partition of n. However, when the order is relevant, we must consider *compositions* of n.

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